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Analysis of pulled axisymmetric membranes with wrinkling

Avinoam Libai [a], Dan Givoli [b,*]

[a] Department of Aerospace Engineering, Technion—Israel Institute of Technology, Haifa 32000, Israel
[b] Department of Aerospace Engineering and Asher Center for Space Research, Technion—Israel Institute of Technology, Haifa 32000, Israel

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Abstract

The nonlinear behavior of an axisymmetric hyperelastic membrane subjected to pulling forces is analyzed. The membrane is considered to be ideal in the sense that it cannot carry compressive stress resultants. If the membrane has a positive initial Gaussian curvature, the pulling gives rise to wrinkles which form over parts of the surface. The full nonlinear equations governing the membrane behavior in the doubly tense and in the wrinkled regions are formulated, and then solved using a numerical integration procedure. Solutions for various examples are presented, with Hookean and neo-Hookean constitutive behavior. These include a few examples of wrinkled membranes with positive initial Gaussian curvatures, and one example of a membrane with a negative initial Gaussian curvature, where no wrinkles are formed.

Keywords: Wrinkles; Ideal membrane; Membrane shell; Tension field

1. Introduction

Very thin elastic membranes, called ideal or true membranes, will *wrinkle* rather than support compressive stresses. Often this results in the formation of wrinkled *tension fields* over portions of the deformed surface. These fields, in which the principal stress resultants are everywhere nonnegative, provide a mechanism for carrying the imposed loads. Within a tension field, the crests and troughs of the wrinkles are parallel to the direction of principal tension, and the stress resultant in the direction normal to them (in the tangent plane) vanishes. This behavior is observed in various practical applications of thin curved membranes, like parachutes, airbags and inflatable structures.

Wu (1978) has given the first complete treatment of wrinkling in nonlinearly elastic membranes of revolution. Zak (1982) extended the theory to wrinkling of films of arbitrary shape. Steigmann and Pipkin (1989), Steigmann (1990), and Li and Steigmann (1995a,b) have made extensive theoretical studies of the behavior of wrinkled and partly wrinkled membranes in various cases, including that of pressurized

* Corresponding author. Present address: Department of Mathematics, Naval Postgraduate School, Code MA/Gd, 1141 Cunningham Road #355, Monterey, CA 93943, USA. Tel.: +972-4-829-3814/+1-831-656-2758; fax: +972-4-823-1848/+1-831-656-2355.
E-mail addresses: givolid@aerodyne.technion.ac.il, dgivoli@nps.navy.mil (D. Givoli).

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spherical and toroidal membranes. Libai (1990) presented a complete theoretical analysis of the transition zone between the doubly tense and wrinkled regions for pulled spherical membranes. Tait et al. (1996) and Tait and Connor (1997) solved wrinkling problems for cylindrical membranes. Roddeman et al. (1987), Gwan and Man (1992), Chiu et al. (1993) and Muttin (1996) developed finite element methods for the solution of partly wrinkled membranes. See Libai and Simmonds (1998, Chapters 5 and 7) for a detailed exposition on the subject and additional references.

The basic idea in the theory of wrinkled membranes is to avoid studying the wrinkled region in detail by replacing it with a smoothed-out *pseudo-surface*. This pseudo-surface must be in equilibrium, and the minimum principal stress resultant must vanish on it. Obviously, the stretch of the pseudo-surface in the direction of the zero minimum principal stress resultant is ‘‘non-physical,’’ namely is not equal to that of the actual wrinkled surface. This simplification leads to the theory of tension fields which the references mentioned above are based on.

In this paper we consider the nonlinear behavior of an axisymmetric hyperelastic membrane subjected to a pulling force. The membrane is attached to two rigid plates along its two edges C and E, as shown in Fig. 1(a). Without loss of generality, we assume that the bottom plate is fixed, whereas the top plate is pulled upwards. During the pull, the upper plate remains parallel to the lower plate and does not rotate with respect to it.

There is a fundamental difference in the response to pulling between membranes with surface of positive Gaussian curvature and those with surface of negative curvature.

The behavior of a membrane with positive Gaussian curvature (see Fig. 1). In this case, the application of a slight pulling force $P = 0^+$ to the upper plate (plate E in Fig. 1(a)) results in the formation of meridional wrinkles, and the membrane surface becomes a ‘‘wrinkled cone;’’ see Fig. 1(b). The formation of wrinkles is due to the fact that the circumferential stress resultants, which would have existed in such a membrane, must be negative, and are replaced by the wrinkle field. In addition, the requirements of equilibrium in the

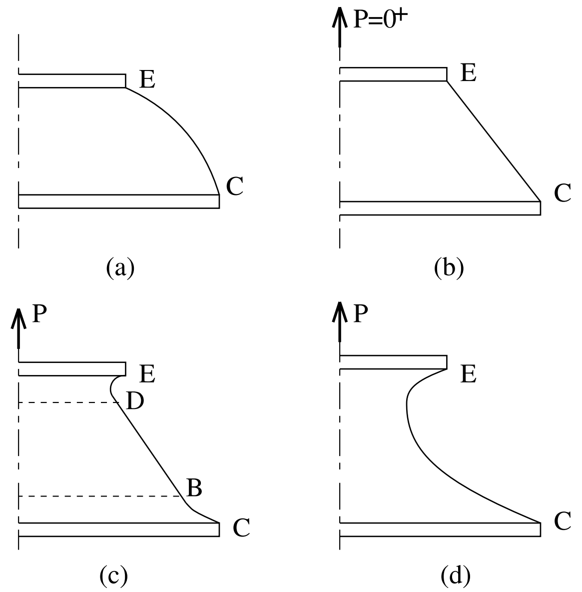

Fig. 1. An axisymmetric elastic membrane subjected to a pulling force: (a) the unloaded membrane; (b) the wrinkled cone generated by a slight pulling force; (c) the wrinkled region (BD) and the two doubly tense regions (CB and ED) formed by a medium-sized pulling force; (d) the all-tense surface generated by a large pulling force.

direction of the normal to the membrane demand that the curvature of the meridional lines be zero. Thus, the meridional wrinkle field takes on the shape of a cone.

In the vast majority of materials for membranes, a Poisson-like effect exists, such that a tensile stress in one direction is accompanied by contraction (negative strain) in the directions normal to it. If this contraction were prevented, say, along a boundary, then positive stresses would form along it and in its immediate neighborhood. The rigid plates at the upper and lower edges of the membrane prevent the transverse contraction, which would have occurred due to the pulling, and, thus, positive circumferential stresses accompany the positive pulling stresses. A biaxial state of stress is, thus, formed at and near the edges. The size of the edge zones is $o(E)$, where E is a typical meridional strain (see Libai, 1990). The entire effect is strongly nonlinear.

An increase in the pulling force leads to a corresponding increase in E, so that the edge zones deteriorate. Thus, increasingly larger doubly tense regions are formed near the boundaries, where both the meridional and tangential (hoop) stress resultants are positive, while the central region remains wrinkled, as illustrated in Fig. 1(c). The meridional curvature of the surface in the doubly tense regions is always negative (see, e.g., Libai and Simmonds, 1998). A further increase in the force causes the central wrinkled region to shrink, until, finally, a tensile *biaxial* state of stress exists over the entire surface, with the complete disappearance of the wrinkled cone; see Fig. 1(d).

For the special case of a symmetric "spherical barrel" see Libai (1990). In the present paper more general axisymmetric membranes are studied.

The behavior of a membrane with negative Gaussian curvature. Here, a tensile biaxial state of stress is formed immediately when the pulling starts. There are no wrinkles, no wrinkled cone and no edge zones. The deformation proceeds uneventfully through its linear and nonlinear phases, until failure eventually occurs. Understandingly, most of the examples in the present paper are of membranes with surfaces having positive Gaussian curvatures. However, our last example concerns the case of a membrane with a surface of negative Gaussian curvature. See the last paragraph of Section 4, and Fig. 16.

In Section 2, we present the full nonlinear equations governing the membrane behavior in the doubly tense and in the wrinkled regions. In Section 3 we show how these equations can be solved by using a numerical integration procedure. We then present numerical results of various example problems in Section 4, for Hookean and for neo-Hookean constitutive behavior. We conclude with some remarks in Section 5.

2. Theory of pulled membranes with wrinkling

We consider a thin curved axisymmetric hyperelastic membrane with a positive Gaussian curvature subjected to an axial pulling force P, as illustrated in Fig. 1(a). We start with some notation. The surface of the unloaded membrane is described by the function $r = r(z)$, where r and z are the radial and axial coordinates, respectively. The arclength coordinate along the meridian is denoted by s. We let $\bar{r}$, $\bar{z}$ and $\bar{s}$ denote, respectively, the radial, axial and arclength locations of a material point in the deformed configuration. Thus, the surface of the deformed membrane is described by the function $\bar{r} = \bar{r}(\bar{z})$. We also let $\bar{\phi}$ be the angle between the axial direction and the normal to the deformed surface, N_s and N_θ be the meridional and tangential stress resultants per unit undeformed length, and $\overline{N}_s$ and $\overline{N}_\theta$ be the meridional and tangential stress resultants per unit deformed length. The relations among the stress resultants are given by the equations

$$rN_s = \bar{r}\overline{N}_s, \qquad \mathrm{d}s N_\theta = \mathrm{d}\bar{s}\,\overline{N}_\theta. \tag{1}$$

Finally, we let λ_s and λ_θ be the principal stretches, which are related to the geometry through

$$\lambda_s = \frac{d\bar{s}}{ds}, \qquad \lambda_\theta = \frac{\bar{r}}{r}. \tag{2}$$

Consideration of the *axial equilibrium* of a portion of the membrane which lies on one side of a circumferential cross section yields

$$\overline{N}_s \sin\bar{\phi} = \frac{P}{2\pi\bar{r}}, \tag{3}$$

which gives, using Eq. (1),

$$N_s = \frac{P}{2\pi r \sin\bar{\phi}}. \tag{4}$$

This equation holds in the entire membrane.

In the doubly tense regions, we also consider the *normal equilibrium* of an element, which yields the equation

$$\frac{\overline{N}_\theta}{\bar{\rho}_\theta} + \frac{\overline{N}_s}{\bar{\rho}_s} = 0. \tag{5}$$

Here, $\bar{\rho}_s$ and $\bar{\rho}_\theta$ are the meridional and tangential radii of curvature of the deformed membrane, which satisfy

$$\frac{1}{\bar{\rho}_s} = -\bar{\phi}_{,\bar{s}}, \qquad \bar{\rho}_\theta = \frac{\bar{r}}{\sin\bar{\phi}}. \tag{6}$$

In Eq. (6) and elsewhere, a comma indicates differentiation. We use Eqs. (1), (5) and (6) to obtain

$$N_\theta = \frac{rN_s\bar{\phi}_{,s}}{\sin\bar{\phi}}. \tag{7}$$

Substituting N_s from Eq. (4) yields

$$N_\theta = -\frac{P}{2\pi}(\cot\bar{\phi})_{,s}. \tag{8}$$

This equation is valid in the doubly tense regions, whereas in the wrinkled region we simply have $N_\theta = 0$. It should be noted that this equation as well as Eq. (4) are the same as Eq. (4.13) derived by Roxburgh et al. (1995) for wrinkled annular membranes in the absence of azimuthal shearing.

The *compatibility relation* simply relates $\bar{r}$, $\bar{s}$ and $\bar{\phi}$ through

$$\frac{d\bar{r}}{d\bar{s}} = -\cos\bar{\phi}. \tag{9}$$

The minus sign in Eq. (9) comes from the fact that in the region where the cosine is positive, namely where $\bar{\phi} < \pi/2$ (as in the segment CB shown in Fig. 1(c)), $\bar{r}$ is a decreasing function of $\bar{s}$. In the doubly tense regions, λ_s and λ_θ are given by Eq. (2). In the wrinkled "cone region" $\bar{\phi}$ is constant, and λ_θ is nonphysical.

The *constitutive relations* complete the set of differential equations governing the membrane's behavior. It is assumed that the membrane possesses a strain energy function $W(\lambda_s, \lambda_\theta)$ per unit area of the undeformed surface, such that

$$N_s = W_{,\lambda_s}, \qquad N_\theta = W_{,\lambda_\theta} \tag{10}$$

in the doubly tense regions. In the wrinkled region, N_θ must be zero, and the second of Eq. (10) furnishes then a relation between λ_s and λ_θ. Thus λ_θ can be eliminated from the first of Eq. (10), and this results in a modified unidirectional constitutive relation for N_s:

$$N_s = f(\lambda_s, \lambda_\theta(\lambda_s)) \equiv f^*(\lambda_s). \tag{11}$$

This has been formalized by Pipkin (1986) who defined a "relaxed" strain energy function to cover the behavior of membranes in both tense and wrinkled regions. The concept has been used by Steigmann and Pipkin (1989) and Steigmann (1990) for curved membranes, as well as by Roxburgh (1995), Roxburgh et al. (1995) and Haughton and McKay (1995) for annular membranes.

We consider two specific examples for the hyperelastic material behavior given by Eqs. (10) and (11). The first example is that of a Hookean (linear elastic) material. Taking for simplicity an isotropic membrane, the strain energy function is quadratic:

$$W(\lambda_s, \lambda_\theta) = \frac{Eh}{1-v^2}\left(\frac{1}{2}(\lambda_s^2 + \lambda_\theta^2) + v\lambda_s\lambda_\theta - (1+v)(\lambda_s + \lambda_\theta - 1)\right). \tag{12}$$

Here h is the membrane thickness, E is Young's modulus and v is Poisson's ratio. With this strain energy function, the relations (10) in the doubly tense regions become

$$N_s = \frac{Eh}{1-v^2}(\lambda_s + v\lambda_\theta) - \frac{Eh}{1-v}, \tag{13}$$

$$N_\theta = \frac{Eh}{1-v^2}(\lambda_\theta + v\lambda_s) - \frac{Eh}{1-v}, \tag{14}$$

whereas the modified constitutive law (11) in the wrinkled region becomes

$$N_s = Eh(\lambda_s - 1). \tag{15}$$

The second example is that of a neo-Hookean material. In this case

$$W(\lambda_s, \lambda_\theta) = Ch\left(\lambda_s^2 + \lambda_\theta^2 + (\lambda_s\lambda_\theta)^{-2} - 3\right), \tag{16}$$

where C is a material constant. In this case the relations (10) in the doubly tense regions become

$$N_s = 2Ch(\lambda_s - \lambda_s^{-3}\lambda_\theta^{-2}), \tag{17}$$

$$N_\theta = 2Ch(\lambda_\theta - \lambda_s^{-2}\lambda_\theta^{-3}), \tag{18}$$

and the modified constitutive law (11) in the wrinkled region becomes

$$N_s = 2Ch(\lambda_s - \lambda_s^{-2}). \tag{19}$$

As we shall see in the next section, the numerical scheme to be employed requires the "inversion" of the constitutive equations (13), (15), (17) and (19), so that in the doubly tense regions λ_s is expressed explicitly in terms of N_s and λ_θ, and in the wrinkled region λ_s is expressed as a function of N_s. In the Hookean case this inversion is trivial, but in the neo-Hookean case it involves the solution of a quartic equation and a cubic equation. We now show how to perform this inversion in the neo-Hookean case.

We define the two non-dimensional parameters

$$\epsilon = \frac{N_s}{2Ch}, \qquad \alpha = \lambda_\theta^{-2}. \tag{20}$$

Then, in the wrinkled region, Eq. (19) can be written as

$$\lambda_s^3 - \epsilon\lambda_s^2 - 1 = 0. \tag{21}$$

This cubic equation can be solved exactly. Setting

$$q = 0.5 + (\epsilon/3)^3, \qquad p = \sqrt{q - 0.25}, \tag{22}$$

the solution is

$$\lambda_s = (q + p)^{1/3} + (q - p)^{1/3} + \epsilon/3. \tag{23}$$

In the doubly tense regions, Eq. (17) can be written as

$$\lambda_s^4 - \epsilon\lambda_s^3 - \alpha = 0. \tag{24}$$

This quartic equation can also be solved exactly, but the exact solution is too cumbersome to be of practical use. Instead, we propose to solve Eq. (24) either numerically or asymptotically. The asymptotic solution is obtained by expanding λ_s in a perturbation series in ϵ:

$$\lambda_s = a_0 + a_1\epsilon + a_2\epsilon^2 + \cdots. \tag{25}$$

We substitute this expansion in Eq. (24), expand the resulting expression in powers of ϵ, and require each coefficient in this expansion to vanish. We have done this up to fourth order and obtained

$$\lambda_s = \alpha^{1/4} + 0.25\epsilon + 0.09375\alpha^{-1/4}\epsilon^2 + 0.03125\alpha^{-1/2}\epsilon^3 + 0.00732\alpha^{-3/4}\epsilon^4 + O(\epsilon^5). \tag{26}$$

It can be shown that this asymptotic series converges if and only if

$$\epsilon < (4/3)\alpha^{1/4}. \tag{27}$$

When this condition is satisfied, numerical tests show that the series converges very quickly. For example, if $\alpha = 1$ and $\epsilon = 0.5$, the residual obtained when the fourth-order solution (26) is substituted into the quartic equation (24) is about 10^{-4}. If the condition (27) is not satisfied, the quartic equation can be solved numerically using a standard root finder.

The governing differential equations given above must be accompanied by *boundary conditions*. The external boundary condition, at the edge points C and E (see Fig. 1), is $\bar{r} = r$ or, equivalently,

$$\lambda_\theta = 1 \quad \text{at C and E.} \tag{28}$$

Interface boundary conditions, at the points B and D separating between the doubly tense and wrinkled zones (see Fig. 1(c)) are

$$N_\theta = 0, \quad \bar{\phi} \text{ is continuous at B and D.} \tag{29}$$

From Eqs. (4) and (15) or (19) we conclude that N_s and λ_s are also continuous at the interfaces. This completes the statement of the problem.

3. Numerical procedure

The system of equations and boundary conditions given in the previous section constitute a nonlinear two-point boundary value problem. We now present a numerical solution scheme, based on a shooting technique, to approximately integrate the equations along the meridian. (The use of such a technique was suggested to the first author by Steigmann; see also Li and Steigmann, 1995a,b.)

The numerical scheme is outlined as follows. The value of the pulling force P is given. Also given is the function $r(z)$ describing the shape of the unloaded membrane, where z ranges from 0 to the height of the undeformed membrane Z. We divide the z axis into M small intervals $[z_{m-1}, z_m]$ for $m = 0, \ldots, M$. For simplicity we take the intervals to be of uniform size $\Delta z \equiv z_m - z_{m-1} = Z/M$. We start from the lower edge of the membrane, i.e., point C in Fig. 1. To initiate the integration process, we first guess the value of the angle $\bar{\phi}$ at point C. Then we integrate the equations numerically as will be described shortly, until we reach the other edge, i.e., point E. We check if the boundary condition (28) is satisfied at point E to within a desired precision; if it is—the solution process has terminated. If not, the guessed value of $\bar{\phi}$ at point C is adjusted, and the integration starts all over again from point C. This process is repeated until the boundary condition (28) is satisfied at point E.

The scheme is now described in full detail. The value of a variable v at location z_m is denoted v_m.

3.1. Zone I: the lower doubly tense zone

(a) Choose an initial value $\bar{\phi}_0$ for the angle $\bar{\phi}$ at point C.
(b) Set $m = 0$, $\bar{r}_0 = r_0 = r(z_0)$, $(\lambda_\theta)_0 = 1$, $s_0 = \bar{s}_0 = 0$, and $z_0 = \bar{z}_0 = 0$.
(c) Use the equilibrium Eq. (4) to calculate

$$(N_s)_m = P/(2\pi r_m \sin\bar{\phi}_m). \tag{30}$$

(d) Use the constitutive equation (13) or (17) to calculate $(\lambda_s)_m$ from $(N_s)_m$ and $(\lambda_\theta)_m$. (In the neo-Hookean case use Eq. (26), provided that Eq. (27) is satisfied.)
(e) Use the constitutive equation (14) or (18) to calculate $(N_\theta)_m$ from $(\lambda_s)_m$ and $(\lambda_\theta)_m$.
(f) Check if $(N_\theta)_m \leqslant 0$. If yes—point B is reached (see Fig. 1(c)); skip to step 2.
(g) Calculate

$$z_{m+1} = z_m + \Delta z, \qquad \Delta r_m = r_{m+1} - r_m, \tag{31}$$

$$\Delta s_m = \sqrt{(\Delta r_m)^2 + (\Delta z)^2}, \quad s_{m+1} = s_m + \Delta s_m. \tag{32}$$

(h) Integrate the equilibrium Eq. (8) to calculate

$$\bar{\phi}_{m+1} = \cot^{-1}[-(2\pi/P)\Delta s_m (N_\theta)_m + \cot\bar{\phi}_m]. \tag{33}$$

(i) Use Eq. (2) to calculate

$$\Delta\bar{s}_m = \Delta s_m (\lambda_s)_m, \qquad \bar{s}_{m+1} = \bar{s}_m + \Delta\bar{s}_m. \tag{34}$$

(j) Use the compatibility equation (9) to calculate

$$\Delta\bar{r}_m = -\Delta\bar{s}_m \cos\bar{\phi}_{m+1}, \qquad \bar{r}_{m+1} = \bar{r}_m + \Delta\bar{r}_m. \tag{35}$$

(k) Use Eq. (2) to calculate

$$(\lambda_\theta)_{m+1} = \bar{r}_{m+1}/r_{m+1}. \tag{36}$$

(l) Calculate

$$\Delta\bar{z}_m = \sqrt{(\Delta\bar{s}_m)^2 - (\Delta\bar{r}_m)^2}, \qquad \bar{z}_{m+1} = \bar{z}_m + \Delta\bar{z}_m. \tag{37}$$

(m) Set $m \leftarrow m + 1$ and return to sub-step (c).

3.2. Zone II: the wrinkled zone

(a) Calculate

$$z_{m+1} = z_m + \Delta z, \quad \bar{\phi}_{m+1} = \bar{\phi}_m, \quad \Delta r_m = r_{m+1} - r_m, \tag{38}$$

$$\Delta s_m = \sqrt{(\Delta r_m)^2 + (\Delta z)^2}, \quad s_{m+1} = s_m + \Delta s_m, \quad (N_\theta)_{m+1} = 0. \tag{39}$$

(b) Use the equilibrium Eq. (4) to calculate

$$(N_s)_{m+1} = P/(2\pi r_{m+1} \sin\bar{\phi}_{m+1}). \tag{40}$$

(c) Use the constitutive equation (15) or (19) to calculate $(\lambda_s)_{m+1}$ from $(N_s)_{m+1}$. (In the neo-Hookean case use Eq. (23).)
(d) Use Eq. (2) to calculate

$$\Delta\bar{s}_m = \Delta s_m(\lambda_s)_{m+1}, \qquad \bar{s}_{m+1} = \bar{s}_m + \Delta\bar{s}_m. \tag{41}$$

(e) Use the compatibility equation (9) to calculate

$$\Delta\bar{r}_m = -\Delta\bar{s}_m \cos\bar{\phi}_{m+1}, \qquad \bar{r}_{m+1} = \bar{r}_m + \Delta\bar{r}_m. \tag{42}$$

(f) Calculate

$$\Delta\bar{z}_m = \sqrt{(\Delta\bar{s}_m)^2 - (\Delta\bar{r}_m)^2}, \qquad \bar{z}_{m+1} = \bar{z}_m + \Delta\bar{z}_m. \tag{43}$$

(g) Use Eq. (2) to calculate a "non-physical" value of λ_θ:

$$(\lambda_\theta^{\mathrm{NP}})_{m+1} = \bar{r}_{m+1}/r_{m+1}. \tag{44}$$

(h) Use the constitutive equation (14) or (18) to calculate a non-physical value of $(N_\theta)_{m+1}$, denoted $(N_\theta^{\mathrm{NP}})_{m+1}$, from $(\lambda_s)_{m+1}$ and $(\lambda_\theta^{\mathrm{NP}})_{m+1}$.
(i) Check if $(N_\theta^{\mathrm{NP}})_{m+1} \geqslant 0$. If yes—point D is reached (see Fig. 1(c)); skip to step 3.
(j) Set $m \leftarrow m+1$ and return to sub-step (a).

3.3. Zone III: the upper doubly tense zone

(a) Repeat sub-steps (c)–(e) and (g)–(m) of step 1. Continue the integration process till the last step $m = M$, namely till $z_m = Z$.
(b) Print out the value of λ_θ at the edge point D, $(\lambda_\theta)_M$. Calculate $\mu = |1 - (\lambda_\theta)_M|$. If $\mu < \delta$, where δ is a predetermined small tolerance (say, $\delta = 10^{-4}$), then the solution process has ended. Otherwise, adjust the value of $\bar{\phi}_0$ (cf. step 1, sub-step (a)) and start the whole process again.

Numerical experiments show that the adjustment of the value of $\bar{\phi}_0$ is easy in practice. The "output" $(\lambda_\theta)_M$ turns out to be a monotonely increasing function of the "input" $\bar{\phi}_0$, so by using a bisection-type technique the correct value of $\bar{\phi}_0$ can be found after a few trials. Numerical experiments also show that the convergence of the results as the number of subintervals M increases is fast. Moreover, the entire calculation is explicit and therefore very efficient. For example, we have used $M = 10{,}000$ for the problems presented in the next section, and the running time in each case on a personal workstation was a few seconds.

4. Examples

We consider an axisymmetric membrane whose undeformed shape is *parabolic*. More precisely, the function $r(z)$ is the parabola

$$r(z) = az^2 + bz + 2. \tag{45}$$

In all cases the height of the membrane in the undeformed state is 2, namely z ranges from 0 to 2. Various values will be chosen for the parameters a and b in (45). Fig. 2 shows some particular undeformed shapes to

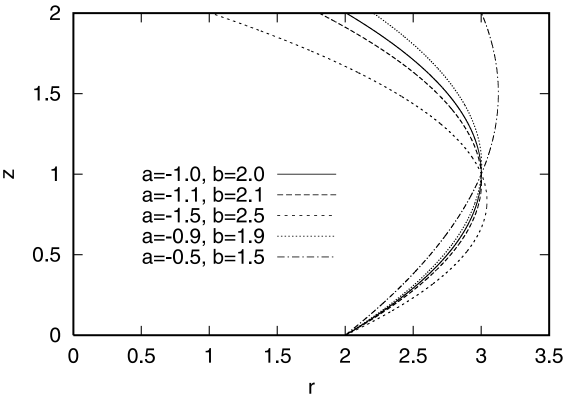

Fig. 2. Undeformed parabolic membranes that are considered as examples.

be considered below. In the cases where Hookean constitutive relations are used we set $Eh = 1$ and $\nu = 0.5$. In the neo-Hookean case we set $Ch = 1$ and $\nu = 0.5$. In all cases we take 10,000 steps in the z-direction.

We start by considering a membrane with a *symmetric* meridian, whose parabola parameters are $a = -1$ and $b = 2$ (see Fig. 2), and with a Hookean material behavior. We apply a force of magnitude $P = 10$. By trial and error (see end of Section 3) we find that the correct initial angle is $\bar{\phi}_0 = 84.55°$. The deformed shape $\bar{r}(\bar{z})$ is shown in Fig. 3. The two interface points separating between the doubly tense and wrinkled regions are marked on the meridian. The wrinkled surface is cylindrical in this case due to the symmetry. Note that the scaling of the $\bar{r}$-axis, here and in other figures, is taken, for better visualization, to be different than that of the $\bar{z}$-axis; thus angles cannot be measured on the graph. We recall that symmetric membrane problems are simpler than non-symmetric ones in that they can be solved analytically; see Libai (1990).

Next we consider a slightly asymmetric membrane, with $a = -1.1$ and $b = 2.1$ (see Fig. 2). Here the upper radius of the undeformed membrane is 10% smaller than the lower radius. The material behavior is still Hookean. Again we set $P = 10$, and we find that $\bar{\phi}_0 = 82.46°$. The deformed shape and transition points are shown in Fig. 4. We choose this specific problem as a prototype and present the results in more detail. Fig. 5 shows the meridional angle $\bar{\phi}$ (in degrees) as a function of the axial coordinate $\bar{z}$. The wrinkled zone is characterized by a constant angle. Fig. 6 shows the distribution of the two stress resultants N_s and

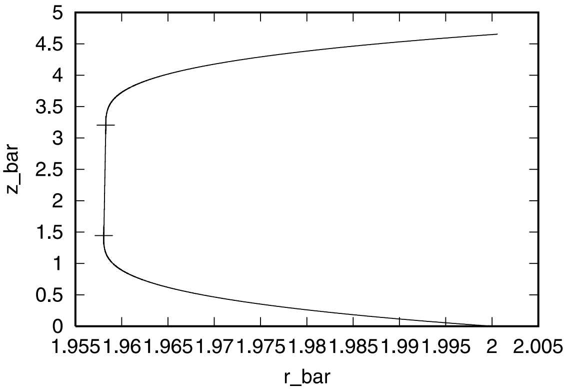

Fig. 3. Membrane with $a = -1$ and $b = 2$ pulled by a force $P = 10$: deformed shape.

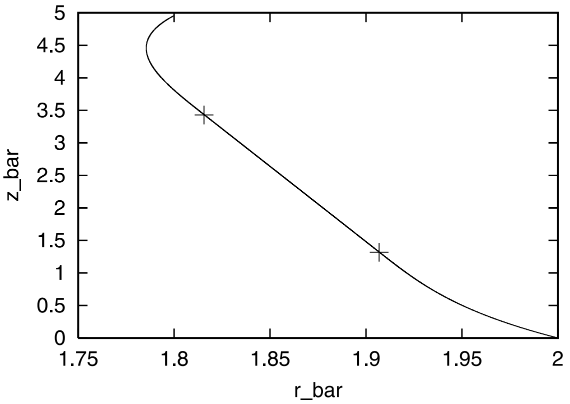

Fig. 4. Membrane with $a = -1.1$ and $b = 2.1$ pulled by a force $P = 10$: deformed shape.

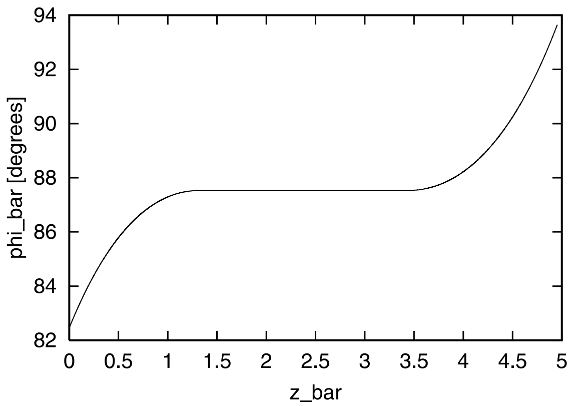

Fig. 5. Membrane with $a = -1.1$ and $b = 2.1$ pulled by a force $P = 10$: meridional angle $\bar{\phi}$ (in degrees) as a function of the axial coordinate $\bar{z}$.

N_θ, while Fig. 7 shows the distribution of the principal stretches λ_s and λ_θ. Note that $N_\theta \equiv 0$ in the wrinkled zone, and that λ_θ is not physical there. Also, λ_s is seen to have a "kink" at the two transition points, namely its rate of change suffers a discontinuity there.

The reason for the kinks lies in the fact that the elastic moduli suffer jumps across the transition points. Compare, for example, Eq. (13) with Eq. (15) or Eq. (17) with Eq. (19). See also Li and Steigmann (1995a).

Now we repeat the solution of this problem while we vary the magnitude of the pulling force P, starting with the very small positive value $P = 1$ and increasing it gradually till $P = 50$. Fig. 8 shows the deformed shapes of the membrane for six different values of P.

For $P = 1$ the deformed surface is almost conical, and except for very small doubly tense regions near the two edges the whole membrane is wrinkled. With $P = 5$ the doubly tense regions become slightly larger, and for $P = 10$ we obtain three distinct regions as in Fig. 4. The wrinkled region continues to shrink as we increase P, until, with $P = 15$, it almost totally vanishes. The two very close transition points are shown on the $P = 15$ curve in Fig. 8. For still larger values of P, the whole membrane becomes doubly tense, as expected. Thus, there is no wrinkling in the surfaces shown in Fig. 8 for $P = 30$ and 50.

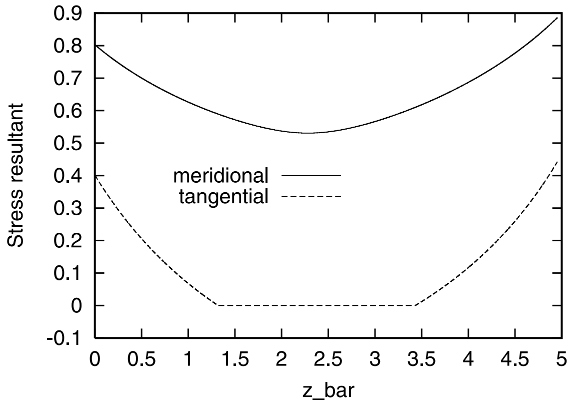

Fig. 6. Membrane with $a = -1.1$ and $b = 2.1$ pulled by a force $P = 10$: stress resultants N_s and N_θ as a function of the axial coordinate $\bar{z}$.

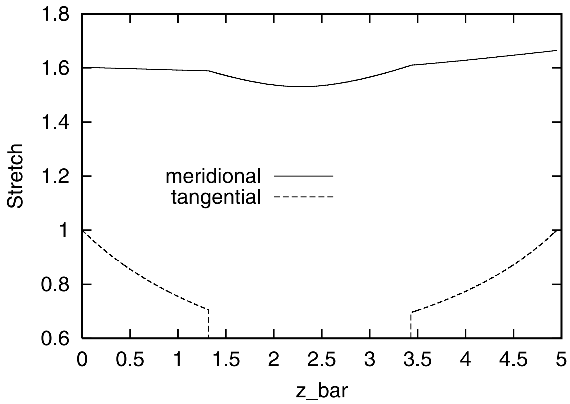

Fig. 7. Membrane with $a = -1.1$ and $b = 2.1$ pulled by a force $P = 10$: stretches λ_s and λ_θ as a function of the axial coordinate $\bar{z}$.

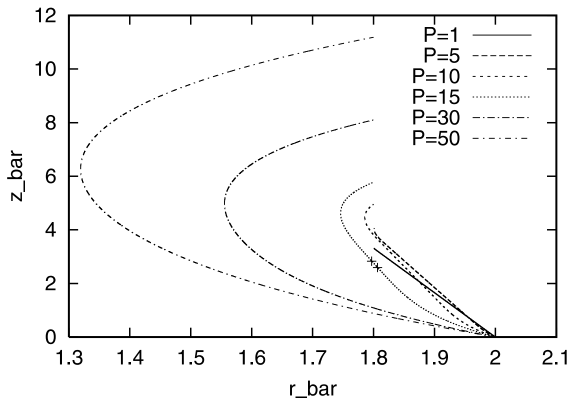

Fig. 8. Membrane with $a = -1.1$ and $b = 2.1$: deformed shapes for different values of the pulling force P.

Now we consider the same undeformed membrane, but with a neo-Hookean material law. We set $P = 30$, and we find $\bar{\phi}_0 = 84.53°$. Figs. 9–12 are the counterparts of Figs. 4–7 that have been obtained in the Hookean case. By comparing the results obtained in the Hookean and neo-Hookean cases, it is apparent that although the results are clearly different quantitatively, the general response of the membrane is similar.

We now return to the Hookean case and remain with it until the end of this section, while we vary the geometry of the unloaded membrane. We also fix the pulling force at $P = 10$. First, we increase the amount of asymmetry of the meridian, and set $a = -1.5$ and $b = 2.5$ (see Fig. 2). We find that $\bar{\phi}_0 = 76.18°$. The deformed shape and transition points are shown in Fig. 13. By comparing Figs. 4 and 13 it is evident that increasing the asymmetry in the shape of the undeformed membrane causes the curvature of the doubly tense regions to decrease.

Now we set $a = -0.9$ and 1.9 (see Fig. 2). This is a slight asymmetry, but in contrast to the case $a = -1.1$, $b = 2.1$ considered before (see Fig. 4), here the upper radius of the undeformed membrane is set 10% *larger* than the lower radius. In this case we find $\bar{\phi}_0 = 86.8°$. Fig. 14 shows the deformed shape. Since the asymmetry is slight, it is not surprising that the shape shown in Fig. 14 is very similar to that shown in Fig. 4, only shifted radially and reflected. When we make the asymmetry strong, by taking $a = -0.5$ and

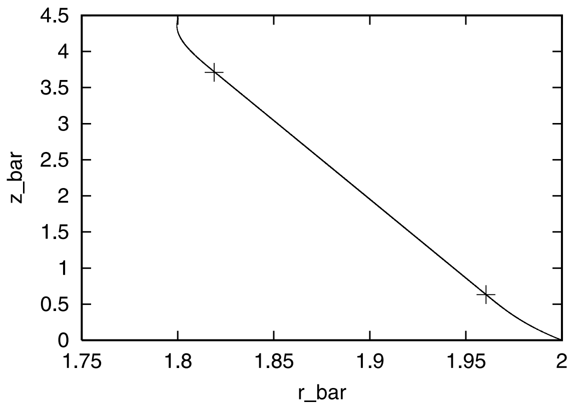

Fig. 9. Membrane with $a = -1.1$ and $b = 2.1$ and a neo-Hookean material law, pulled by a force $P = 30$: deformed shape.

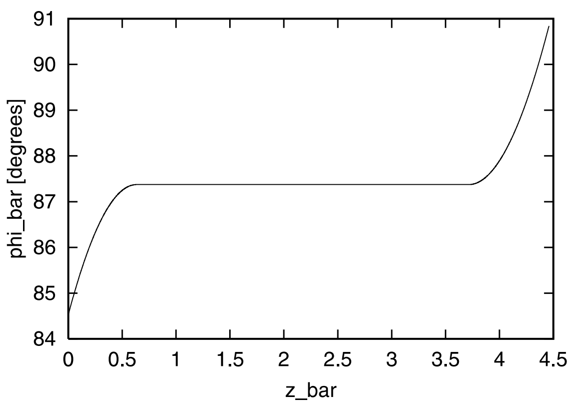

Fig. 10. Membrane with $a = -1.1$ and $b = 2.1$ and a neo-Hookean material law, pulled by a force $P = 30$: meridional angle $\bar{\phi}$ (in degrees) as a function of the axial coordinate $\bar{z}$.

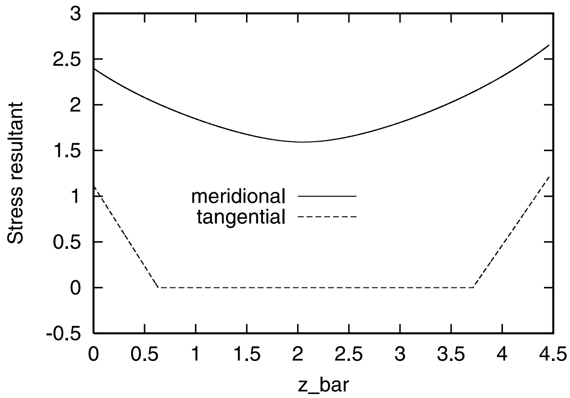

Fig. 11. Membrane with $a = -1.1$ and $b = 2.1$ and a neo-Hookean material law, pulled by a force $P = 30$: stress resultants N_s and N_θ as a function of the axial coordinate $\bar{z}$.

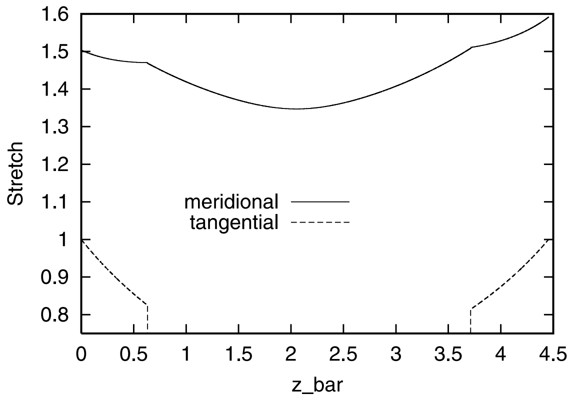

Fig. 12. Membrane with $a = -1.1$ and $b = 2.1$ and a neo-Hookean material law, pulled by a force $P = 30$: stretches λ_s and λ_θ as a function of the axial coordinate $\bar{z}$.

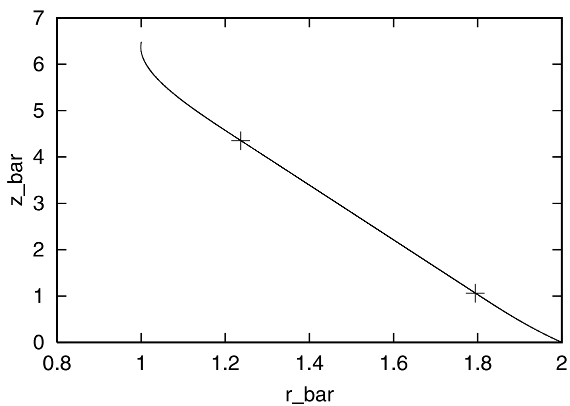

Fig. 13. Membrane with $a = -1.5$ and $b = 2.5$ pulled by a force $P = 10$: deformed shape.

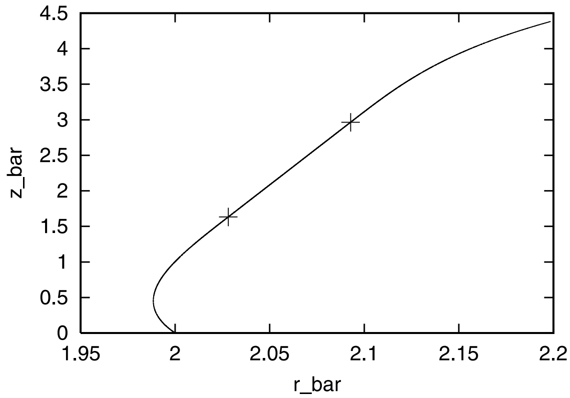

Fig. 14. Membrane with $a = -0.9$ and $b = 1.9$ pulled by a force $P = 10$: deformed shape.

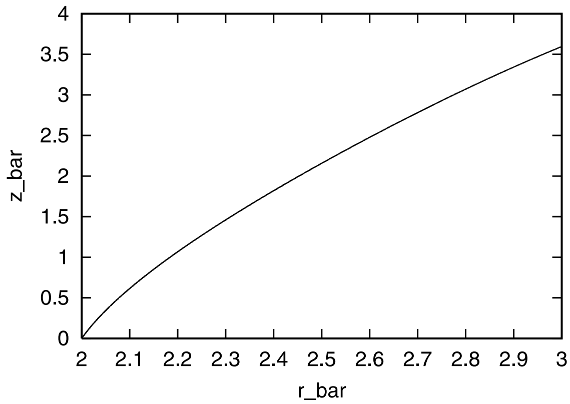

Fig. 15. Membrane with $a = -0.5$ and $b = 1.5$ pulled by a force $P = 10$: deformed shape.

$b = 1.5$ (see Fig. 2), we find that $\bar{\phi}_0 = 96.85°$, and we obtain the deformed shape shown in Fig. 15. Here the entire membrane is doubly tense and there is no wrinkling at all. The hoop stress resultant N_θ decreases from the value 0.40 at the lower edge of the membrane to a minimum value of 0.11 at a point about the center of the meridian, and then increases back to the value 0.29 at the upper edge, thus never becoming compressive.

All the membranes considered thus far had *positive meridional curvature* in the undeformed configuration. As we have shown, in the deformed configuration the curvature is always negative in the doubly tense regions. Our final numerical example is that of a membrane with *negative curvature* in the undeformed configuration. To this end, we set $a = 1$ and $b = -2$, which yield a symmetric undeformed meridian. For $P = 10$ we find that $\bar{\phi}_0 = 58.10°$. Fig. 16 shows the deformed and undeformed shapes of the membrane. Here, again, the whole membrane is doubly tense. In fact, the hoop stress resultant N_θ *increases* from its value at the lower edge (0.47) to a maximum value (0.92) at the center of the meridian, and then back to its minimum value (0.47) at the upper edge. This is expected. In fact, physical reasoning easily leads to the conclusion that in the case of an initial negative meridional curvature there is never any wrinkling. It should be noted that our method of analysis applies to this case with the same ease as to the case of an initial positive curvature.

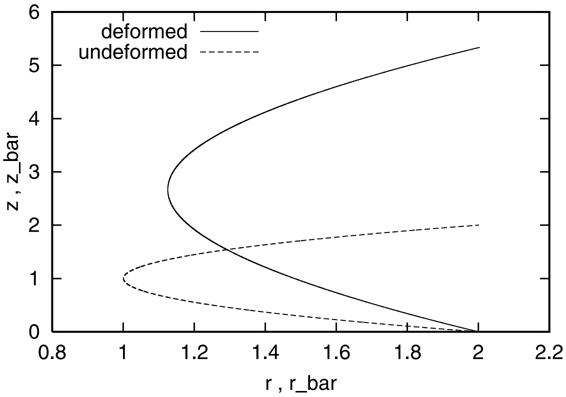

Fig. 16. Membrane with $a = 1$ and $b = -2$ (with a negative undeformed curvature) pulled by a force $P = 10$: deformed and undeformed shapes.

5. Concluding remarks

Partial wrinkling is a prominent feature in the behavior of many nonlinear membranes, especially near boundaries and stiffeners. Many examples exist in biological membranes, inflatable structures and in the limiting behavior (as $h \to 0$) of buckled plates and shells. Therefore, complete solutions to nontrivial problems which are geometrically nonlinear and possibly also materially nonlinear are of interest.

In this paper we have extended the work of Libai (1990) on partly wrinkled membranes of revolution with a symmetric meridian (like the "spherical barrel") to membranes of revolution with a generally shaped meridian. In doing so, we have analyzed the fully nonlinear problem, without introducing any simplifying assumptions like small strains or proximity to a known state.

The solution of the equations has been performed by approximately integrating them using a shooting-type numerical technique. This procedure is quite simple and successful despite the complexity of the problem which involves the highly nonlinear membrane equations and different regions with interfaces which are unknown a priori. In comparison, an attack of the same class of axisymmetric problems by a finite element method would be much more complicated. However, generalizing the numerical approach used here to curved membranes which are not axisymmetric may prove to be difficult.

Acknowledgements

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